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Cooperative Control of Linear Systems with Coupled Constraints via Distributed Model Predictive Control

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Abstract: This paper presents a distributed model predictive control(DMPC) strategy to achieve cooperative tasks like consensus and synchronization of linear systems with coupled constraints. Based on the error of state between each subsystem and the center of its cooperative set, a novel DMPC algorithm is designed with its robust feasibility guaranteed by permitting only one subsystem to optimize at each time instant, while renew other subsystems' plans. Illustrative examples are finally conducted to verify the effectiveness of the DMPC strategy.

Key Words: Distributed model prediction control(DMPC), linear systems, coupled constraints, cooperative, robust feasibility

1 Introduction

Collective behaviors of decentralized and distributed systems, such as consensus and synchronization, have attracted much attention over the last few decades. Vivid examples exist in biology, physics, computer science, automatic control, and distributed cooperative systems, such as swarming/flocking, collaborative robots, underwater boats, unmanned air vehicles, formation control, congestion control in sensor networks, see e.g., [1]-[4]. In contrast to classical control objective like the stabilization of an a priori known set point or following an a priori specified reference trajectory, the objective of consensus and synchronization is that all distributed subsystems agree upon certain quantities of interest [5]-[10].

On the other hand, abundant references in biology literature have presented that almost all living creatures apply the prediction intelligence allowing them to predict the future state of their neighbors and themselves in cooperative work [11, 12]. Examples about this prediction intelligence are vividly reflected in bee swarm formation [11], bio-eyesight systems [12] and so on. Such predictive mechanisms with the capabilities of optimizing some performance criterion and taking constraints into account [13]-[14], has become one of the most useful control strategies in many industrial processes. These significant advantages have attracted many researchers to apply these alluring features in the study of decentralized and distributed networks of interacting systems. As a representative work, Camponogara, E., et al. [15] have focused on the distributed forms of Model predictive control (MPC), in which decision making is distributed among different subsystems to guarantee the stability of the whole closed-loop system. Distributed MPC for the stability of the dynamically decoupled systems has also been considered in [16]-[18].

However, only the stabilization of an a priori known set point has been considered in the previous work, including those mentioned above. There exist few exceptions considering other cooperative tasks like the consensus and synchronization. In [19], Farrai-Trecate et al. have proposed decentralized model predictive control schemes that take into account constraints on the agent's input for agents have a discrete-time single- or double-integrator dynamics and shown that they guarantee consensus under mild assumptions. Consensus is achieved by exploiting geometric properties of the optimal paths. Later, Zhang *et al.* have proposed a MPC strategy to achieve consensus and numerical simulation has verified the performance of consensus in [20]. Furthermore, inspired by some efficient schemes in pinning control, they have presented an improved control strategy with faster convergence speed in [21]. Afterwards, a general framework for DMPC of discrete-time nonlinear systems has been proposed in [22], in which the cooperative tasks like consensus and synchronization are well handled. And all the subsystems optimize in a sequential order at each time instant in order to guarantee the feasibility and convergence to the desired cooperative goal. Moreover, the terminal cost functions and the terminal region is suitably defined and computed in the latter case. Following this line, Zhan and Li [23] have introduced a DMPC weighted average consensus protocol for a continuous-time multi-agent network in the sampled-data setting, and proved such a sampled-data multi-agent system with one-order integrator dynamic asymptotically reaches the weighted-average consensus via the DMPC protocol with both fixed and switching network topologies. Recently, Trodden, P. et al. [24] have developed a cooperative, distributed form of MPC for linear systems subject to persistent, bounded disturbances, where a local agent design hypothetical plans for other agents with the local performance sacrificed to make up the whole. The robust feasibility is guaranteed by permitting only one agent to optimize at each time instant, while freezing other agents' plans. And a key feature that coupled constraint satisfaction is compatible with inter-agent cooperation is also needed.

In short, designing a DMPC strategy with robust feasibility for the cooperative work of linear systems with coupled constraints is still a challengeable work. In this paper, our main contributions include: (a) based on the error of state between each subsystem and the center of its cooperative set, a novel DMPC strategy(Algorithm 1) is designed for lin-

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ear systems' cooperative tasks like consensus and synchronization problems instead of the stabilization of an a priori known set point which most previous work has considered. (b) robust feasibility of the DMPC algorithm is guaranteed by permitting only one subsystem to optimize at each time instant, while renew other subsystems' plans. (c) the consensus and synchronization of the linear systems with arbitrarily selected cooperative sets is illustratively displayed.

The rest of this paper is organized as follows. Section 2 introduces some necessary preliminaries and statement. In Section 3, we propose a DMPC strategy for the cooperative work of linear systems with coupled constraints and prove the robust feasibility of the DMPC algorithm. Illustrative examples showing the performance of our proposed DMPC strategy are presented in Section 4. Finally, conclusions are drawn in Section 5.

2 Preliminaries and Statement

2.1 Notation

We first introduce the following definitions and notations to use throughout this paper. Denote by $\mathbb{R}$ the field of real numbers, and for any $a \in \mathbb{R}$, $|a|$ defines the absolute value of a. For any vector $\boldsymbol{b_i} \in \mathbb{R}^n$, denote $\|\boldsymbol{b_i}\|$ as Euclidean norm. Let $\{\boldsymbol{b_i}\}_{i\in\mathcal{I}}$ indicate the collection of vectors $\boldsymbol{b_i}$ for all i in the index $\mathcal{I}$. Matrix I_n represents the identity matrix with $n-$dimension, and $\otimes$ denotes the Kronecker product. The definition $*(k+t|k)$ denotes the prediction value $*$ at time instant $k+t$ based on the currently available information at time instant k.

2.2 Problem Statement

Consider a network composed of N linear subsystems, each described by

$$\boldsymbol{x_i(k+1)} = (I_T \otimes \boldsymbol{A})\boldsymbol{x_i(k)} + (I_T \otimes \boldsymbol{B})\boldsymbol{u_i(k)} \qquad (1)$$

where $\boldsymbol{x_i} \in \mathcal{X}_i \subseteq \mathbb{R}^{n_i}$ and $\boldsymbol{u_i} \in \mathcal{U}_i \subseteq \mathbb{R}^{m_i}$ denote respectively the state and control of the $i-th$ subsystem, and $i \in \mathcal{I} := \{1, 2, \cdots, N\}$. Define $\boldsymbol{x_i(k)} := [\boldsymbol{x_i(k|k)}^T, \cdots, \boldsymbol{x_i(k+T|k)}^T]^T$ and $\boldsymbol{u_i(k)} := [\boldsymbol{u_i(k|k)}^T, \cdots, \boldsymbol{u_i(k+T-1|k)}^T]^T$ with prediction horizon T. Denote by $\boldsymbol{x}$ and $\boldsymbol{u}$ the state and input of the overall system, i.e., $\boldsymbol{x} := [\boldsymbol{x_1}^T, ..., \boldsymbol{x_N}^T]^T$ and $\boldsymbol{u} := [\boldsymbol{u_1}^T, ..., \boldsymbol{u_N}^T]^T$, and let $\mathcal{X} := \mathcal{X}_1 \times \cdots \times \mathcal{X}_N$ and $\mathcal{U} := \mathcal{U}_1 \times \cdots \times \mathcal{U}_N$.

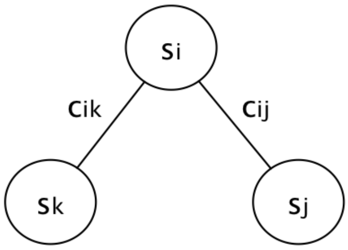


Fig. 1 *Three subsystems with coupled constraints*

Though the linear subsystems (1) are dynamically decoupled, they are coupled with each other via some common constraints c. Thus for each subsystem i, we define subsystem j to be a neighbor of subsystem i if they are subject to common coupled constraints c_{ij}. We give an illustrative example in Fig. 1, where subsystem s_i and s_j are coupled by constraint c_{ij}, and subsystem s_i and s_k are coupled by constraint c_{ik}. Abstractly, each subsystem i can be described as a vertex of a weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, consisting of a set of vertices $\mathcal{V} = \{1, 2, ..., N\}$, edges $\mathcal{E} \subseteq \{(i, j) : i, j \in \mathcal{V}, j \neq i\}$, and a weighted adjacency matrix $\mathcal{A} = [a_{ij}]_{N\times N}$ with nonnegative adjacency elements a_{ij}. We define $a_{ij} = 1$ if $i, j \in \mathcal{V}$, else $a_{ij} = 0$. The neighbor set of subsystem i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ and $|\mathcal{N}_i|$ denotes the number of the neighbors of subsystem i. The cooperative set for subsystem i, consisting of itself and its neighbors, is defined by $\delta_i = \{N_1^i, N_2^i, ..., N_{|\mathcal{N}_i|+1}^i\}$, $N_1^i < N_2^i < \cdots < N_{|\mathcal{N}_i|+1}^i$, $N_j^i \in \mathcal{N}_i \cup i$, $j = 1, 2, ..., |\ \mathcal{N}_i\ | + 1$. And $|\delta_i|$ indicates its cardinality. Since common constraints affect both associated subsystems, we obtain subsystem i is a neighbor of subsystem j if and only if subsystem j is also a neighbor of subsystem i. This also means the graph $\mathcal{G}$ is undirected, i.e, for any $i, j \in \mathcal{V}$, $(i, j) \in \mathcal{E} \iff (j, i) \in \mathcal{E}$. The Laplacian matrix $\mathcal{L}_N = [l_{ij}]_{N\times N}$ associated with $\mathcal{G}$ is defined as, $l_{ij} = -a_{ij}, i \neq j$, and $l_{ii} = \sum_{j=1, j\neq i}^{N} a_{ij}$.

Our target is to design a distributed control law such that the overall network system is asymptotically stable while consensus and synchronization of all the subsystems, defined as $\{\boldsymbol{x_1} = \boldsymbol{x_2} = \cdots = \boldsymbol{x_N}\}$ is achieved. To facilitate understanding, we first give a brief introduce of the common method to obtain optimal MPC input $\boldsymbol{u_i^*(k|k)}$ for subsystem i at instant k.

Following some basic rules from MPC, we compute the prediction states $\boldsymbol{x_i(k+t|k)}, (t = 1, 2, ..., T)$:

$$\begin{aligned} \boldsymbol{x_i(k+t|k)} &= \boldsymbol{Ax_i(k+t-1|k)}, \\ &+ \boldsymbol{Bu_i(k+t-1|k)}, t < H_u \end{aligned} \qquad (2)$$

$$\begin{aligned} \boldsymbol{x_i(k+t|k)} &= \boldsymbol{Ax_i(k+t-1|k)}, \\ &+ \boldsymbol{Bu_i(k+H_u-1|k)}, t \geq H_u \end{aligned} \qquad (3)$$

Here, integer H_u denotes the control horizon with $1 \leq H_u \leq T$. Thus Eqs. (2) and (3) can be rewritten in a compact way as:

$$\boldsymbol{x_i(k)} = \boldsymbol{P_x x_i(k|k)} + \boldsymbol{P_u u_i(k)}$$

with $\boldsymbol{P_x} = col[\boldsymbol{A}, \boldsymbol{A^2}, ..., \boldsymbol{A^T}]$ and $\boldsymbol{P_u} =$

$$\begin{bmatrix} \boldsymbol{B} & & & \\ \boldsymbol{AB} & \ddots & & \\ \vdots & \vdots & \boldsymbol{B} & \\ \boldsymbol{A^{H_u-1}B} & \dots & \boldsymbol{AB} & \boldsymbol{B} \\ \vdots & \dots & \vdots & \vdots \\ \boldsymbol{A^{T-1}B} & \dots & \boldsymbol{A^{T-H_u+1}B} & \sum_{i=0}^{T-H_u} \boldsymbol{A^i B} \end{bmatrix}$$

Then we definite the MPC cost function for agent i as:

$$J_i(k) = \|\boldsymbol{P_x x_i(k|k)} + \boldsymbol{P_u u_i(k)} - \boldsymbol{r_i(k)}\|^2_{\boldsymbol{\Theta_i}} + \|\boldsymbol{u_i(k)}\|^2_{\boldsymbol{\Lambda_i}} \qquad (4)$$

where $\boldsymbol{\Theta_i}$ and $\boldsymbol{\Lambda_i}$ represent associated state-weighted matrix with an appropriate dimension and control-weighted matrix with an appropriate dimension, respectively. $\boldsymbol{r_i(k)}$ denotes the reference signal. Optimizing the MPC cost function by using $\frac{\partial J_i(k)}{\partial \boldsymbol{u_i(k)}}$, we obtain the optimal input sequence : $\boldsymbol{u_i^*(k)} = -(\boldsymbol{P_u^T}\Theta_i\boldsymbol{P_u} + \boldsymbol{\Lambda_i})^{-1}\boldsymbol{P_u^T}\Theta_i(\boldsymbol{P_x x_i(k|k)} -$

$r_i(k)$). Usually, we use the first entry of u_i^* as the actual control signal, i.e., $u_i^*(k|k) = -S \times u_i^*(k)$ with $S :=$

$$\begin{bmatrix} \underbrace{1,0,...,0}_{H_u} & \underbrace{0,0,...,0}_{H_u} & \cdots & \\ \underbrace{0,0,...,0}_{H_u} & \underbrace{1,0,...,0}_{H_u} & \underbrace{0,0,...,0}_{H_u} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \underbrace{0,0,...,0}_{H_u} & \underbrace{0,0,...,0}_{H_u} & \cdots & \underbrace{1,0,...,0}_{H_u} \end{bmatrix}_{m_i \times m_i H_u}$$

Denoting $\varphi_i = S \times (P_u^T \Theta_i P_u + \Lambda_i)^{-1} P_u^T \Theta_i$, we have the optimal MPC input

$$u_i^*(k|k) = -\varphi_i(P_x x_i(k|k) - r_i(k)) \tag{5}$$

where the vector φ_i is independent of $x_i(k|k)$ and thus can be calculated offline.

Remark 1: *By optimizing the MPC cost function (4) associated with prediction and control horizon, we have obtained the optimal MPC input. Inspired by this rolling or receding horizon optimization strategy, we will design the distributed MPC input to better adapt to the MAS (1) afterwards.*

Considering the interactions and coupled constraints between each subsystem and its neighbors, we want to achieve the robust feasibility by DMPC.

For computing convenience, let control horzion H_u equal to prediction horizon T. Then the finite horizon optimization problem for each subsystem i is now formally described as below:

Problem $\mathcal{P}_i$: At instant k,

$$\begin{aligned} &\min_{u_i(k)} J_i(\varepsilon_i(k), u_i(k)) \\ &= \sum_{t=0}^{T-1} (\|\varepsilon_i(k+t|k)\|_{Q_i}^2 + \|u_i(k+t|k)\|_{R_i}^2) \\ &+ \|\varepsilon_i(k+T|k)\|_{P_i}^2 \end{aligned} \tag{6a}$$

with $\varepsilon_i := x_i - \frac{1}{|\delta_i|}\sum_{j\in\delta_i} x_j$ indicating the error of state between each subsystem and the center of its cooperative set.

Subject to

$$\begin{aligned} &x_i(k+t+1|k) \\ &= Ax_i(k+t|k) + Bu_i(k+t|k) \end{aligned} \tag{6b}$$

$$x_i(k+t+1|k) \subseteq \mathcal{X}_i \tag{6c}$$

$$u_i(k+t|k) \subseteq \mathcal{U}_i \tag{6d}$$

$$q_c(x_i(k+t+1|k), \{x_j\}_{j\in\mathcal{N}_i}(k+t+1|k)) \subseteq \mathcal{Q}_c \tag{6e}$$

for all $t \in \{0, 1, \cdots, T-1\}$

The functions q_c and the sets $\mathcal{Q}_c \subseteq \mathbb{R}^{\mathcal{L}_c}$ with $c \in \{1, \cdots, \mathcal{C}\}$ denote respectively the $\mathcal{C}$ coupling outputs and $\mathcal{C}$ coupling constraint sets the systems are confined to.

Similar to the terminal cost and terminal region method in MPC, the crucial assumption to ensure the stability of the overall system is that there exists a admissible control invariant terminal set. We will give the descriptions here.

Assumption 1 *There exists a terminal region $\mathcal{X}^T \in \mathcal{X}$, and terminal control law $u_i^T = k_i^T(x_i, \{x_j\}_{j\in\mathcal{N}_i})$, such that the terminal region $\mathcal{X}^T$ is invariant with respect to the overall closed-up system $x(k+1) = (I_N \otimes A)x(k) + (I_N \otimes B)u^T$ with $u^T = [u_1^{T^T}, \cdots, u_N^{T^T}]^T$. And the following holds for all $x \in \mathcal{X}^T$ and for all $i \in \mathcal{I}$:*

$$k_i^T(x_i, \{x_j\}_{j\in\mathcal{N}_i}) \in \mathcal{U}_i \tag{7a}$$

$$q_c(x_i, \{x_j\}_{j\in\mathcal{N}_i}) \subseteq \mathcal{Q}_c \quad c \in \{1, \cdots, \mathcal{C}\} \tag{7b}$$

$$\sum_{i=1}^{N} \|\varepsilon_i^+\|_{P_i}^2 + \|\varepsilon_i\|_{Q_i}^2 - \|\varepsilon_i\|_{P_i}^2 + \|u_i^T\|_{R_i}^2 \leq 0 \tag{7c}$$

with $\varepsilon_i^+ := \varepsilon_i(k+1+T|k+1)$ at instant k.

In assumption 1, (7a) indicates that the terminal control laws also satisfy the input constraints. (7b) represents the satisfaction of the coupling constraints inside the terminal region. (7c) denotes the decay rate of the terminal energy in the overall system.

Here we choose $u_i^T := k_i^T(x_i, \{x_j\}_{j\in\mathcal{N}_i}) = -k_i\varepsilon_i$ and rewrite the inequality (7c) in a directed way. First define $\varepsilon(k) := [\varepsilon_1(k)^T, \cdots, \varepsilon_N(k)^T]^T$, $x(k) := [x_1(k)^T, \cdots, x_N(k)^T]^T$ and $\Delta := diag\{\frac{1}{|\delta_i|}\}_{N\times N}$. Then we obtain

$$\begin{aligned} &\varepsilon(k+1+T|k+1) \\ &= (\Delta \otimes I_n)(\mathcal{L}_N \otimes I_n)x(k+1+T|k+1) \\ &= (\Delta \otimes I_n)(\mathcal{L}_N \otimes I_n)((I_N \otimes A)x(k+T|k) \\ &+ (I_N \otimes B)u^T(k+T|k)) \\ &= (\Delta \otimes I_n)(\mathcal{L}_N \otimes I_n)((I_N \otimes A)x(k+T|k) \\ &+ (I_N \otimes B)(I_N \otimes k_i)(\Delta \otimes I_n)(\mathcal{L}_N \otimes I_n)x(k+T|k)) \\ &= ((I_N \otimes A) + (\Delta \otimes I_n)(\mathcal{L}_N \otimes I_n) \\ &\times (I_N \otimes B)(I_N \otimes k_i))\varepsilon(k+T|k) \end{aligned} \tag{8}$$

Finally, substitute (8) into (7c) to obtain that:

$$\begin{aligned} &\sum_{i=1}^{N} \|\varepsilon_i^+\|_{P_i}^2 + \|\varepsilon_i\|_{Q_i}^2 - \|\varepsilon_i\|_{P_i}^2 + \|u_i^T\|_{R_i}^2 \\ &= \varepsilon(k+1+T|k+1)^T P \varepsilon(k+1+T|k+1) \\ &+ \varepsilon(k+T|k)^T Q \varepsilon(k+T|k) \\ &- \varepsilon(k+T|k)^T P \varepsilon(k+T|k) \\ &+ \varepsilon(k+T|k)^T (I_N \otimes k_i)^T R (I_N \otimes k_i) \varepsilon(k+T|k) \\ &= \varepsilon(k+T|k)^T ((I_N \otimes A) + (\Delta \otimes I_n)(\mathcal{L}_N \otimes I_n) \\ &\times (I_N \otimes B)(I_N \otimes k_i))^T P((I_N \otimes A) \\ &+ (\Delta \otimes I_n)(\mathcal{L}_N \otimes I_n)(I_N \otimes B)(I_N \otimes k_i))\varepsilon(k+T|k) \\ &+ \varepsilon(k+T|k)^T Q \varepsilon(k+T|k) \\ &- \varepsilon(k+T|k)^T P \varepsilon(k+T|k) \\ &+ \varepsilon(k+T|k)^T (I_N \otimes k_i)^T R (I_N \otimes k_i) \varepsilon(k+T|k) \\ &= \varepsilon(k+T|k)^T (((I_N \otimes A) + (\Delta \otimes I_n)(\mathcal{L}_N \otimes I_n) \end{aligned}$$

$$\times (I_N \otimes \boldsymbol{B})(I_N \otimes \boldsymbol{k_i}))^T P((I_N \otimes \boldsymbol{A}) + (\Delta \otimes I_n)$$
$$\times (\mathcal{L}_N \otimes I_n)(I_N \otimes \boldsymbol{B})(I_N \otimes \boldsymbol{k_i})) - P$$
$$+ (I_N \otimes \boldsymbol{k_i})^T R(I_N \otimes \boldsymbol{k_i}) + Q)\boldsymbol{\varepsilon(k+T|k)}$$
$$= \boldsymbol{\varepsilon(k+T|k)}^T((\widetilde{\boldsymbol{A}} + \widetilde{\mathcal{L}}\widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{K}})^T P(\widetilde{\boldsymbol{A}} + \widetilde{\mathcal{L}}\widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{K}})$$
$$- P + \widetilde{\boldsymbol{K}}^T R\widetilde{\boldsymbol{K}} + Q)\boldsymbol{\varepsilon(k+T|k)} \quad (9)$$

With $\widetilde{\boldsymbol{A}} := (I_N \otimes \boldsymbol{A})$, $\widetilde{\boldsymbol{B}} := (I_N \otimes \boldsymbol{B})$, $\widetilde{\boldsymbol{K}} := (I_N \otimes \boldsymbol{k_i})$, $\widetilde{\mathcal{L}} := (\mathcal{L}_N \otimes I_n)$ and $P := diag\{P_i\}_{N\times N} \otimes I_n$, $Q := diag\{Q_i\}_{N\times N} \otimes I_n$, $R := diag\{R_i\}_{N\times N} \otimes I_n$.

Till now, the inequality (7c) can be rewritten as:

$$\boldsymbol{\varepsilon(k+T|k)}((\widetilde{\boldsymbol{A}} + \widetilde{\mathcal{L}}\widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{K}})^T P(\widetilde{\boldsymbol{A}} + \widetilde{\mathcal{L}}\widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{K}}) - P$$
$$+ \widetilde{\boldsymbol{K}}^T R\widetilde{\boldsymbol{K}} + Q)\boldsymbol{\varepsilon(k+T|k)} \leq 0 \quad (10)$$

LMI tools in Matlab can be used to obtain the appropriate coefficient matrices $\boldsymbol{k_i}$ and P_i. Here, we transform the inequality as the following equivalent LMI with $X := P^{-1} > 0$ and $Y = \widetilde{\boldsymbol{K}}X$ by using standard manuipulations such as the Schur complement and left and right multiplying with P^{-1} (see more details in [25]):

$$\begin{bmatrix} X & X\widetilde{\boldsymbol{A}}^T + Y^T\widetilde{\boldsymbol{B}}^T & XQ^{1/2} & Y^T R^{1/2} \\ \widetilde{\boldsymbol{A}}X + \widetilde{\boldsymbol{B}}Y & X & 0 & 0 \\ Q^{1/2}X & 0 & I & 0 \\ R^{1/2}Y & 0 & 0 & I \end{bmatrix} \geq 0 \quad (11)$$

3 Cooperative control via DMPC

In this section, we will formulate our DMPC strategy. The feasibility and the stability of this DMPC algorithm are also fully analyzed and guaranteed.

The proposed DMPC algorithm is described as follows.

Algorithm 1 *DMPC for a subsystem $i \in \mathcal{I}$*

0) Initialization: set $k = 0$. Wait for a feasible solution $\hat{\boldsymbol{u}}_{\boldsymbol{i}}(\boldsymbol{0})$ with corresponding state sequence $\hat{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{0})$.

1) Sample current state $\boldsymbol{x_i(k)}$.

2) Update plan. if $i_k = k$.

(a) *Receive states $\boldsymbol{x_j(k+t|k)} := \hat{\boldsymbol{x}}_{\boldsymbol{j}}\boldsymbol{(k+t|k-1)}$, $t \in \{0, 1, \cdots, T-1\}$ and $\boldsymbol{x_j(k+T|k)} \in \mathcal{X}^T$ for all $j \in \mathcal{N}_i$ which have not yet calculated their optimal inputs; Or $\boldsymbol{x_j(k+t|k)} := \boldsymbol{x_j^*(k+t|k-1)}$, $t \in \{0, 1, \cdots, T-1\}$ and $\boldsymbol{x_j(k+T|k)} \in \mathcal{X}^T$ for all $j \in \mathcal{N}_i$ which have already calculated their optimal inputs at instant k.*

(b) *Obtain new plan $\boldsymbol{u_i(k)} = \boldsymbol{u_i^*(k)}$ by solving the problem $\mathcal{P}_i$ (2) and the corresponding state by $\boldsymbol{x_i^*(k)}$.*

(c) *Transmit optimal state $\boldsymbol{x_i^*(k)}$ to other subsystems j if $i \in \mathcal{N}_j$.*

else,

(a) *Renew current plan: $\boldsymbol{u_i(k)} = \hat{\boldsymbol{u}}_{\boldsymbol{i}}\boldsymbol{(k)} := \{\boldsymbol{u_i^*(k|k-1)}, \cdots \{\boldsymbol{u_i^*(k+T-2|k-1)}, \boldsymbol{k_i^T}(\boldsymbol{x_i^*(k+T-1|k-1)}, \hat{\boldsymbol{x}}_{\boldsymbol{j}}\boldsymbol{(k+T-1|k-1)})\}$. Obtain the corresponding state $\hat{\boldsymbol{x}}_{\boldsymbol{i}}\boldsymbol{(k)}$.*

(b) *Transmit optimal state $\hat{\boldsymbol{x}}_{\boldsymbol{i}}\boldsymbol{(k)}$ to other subsystems j if $i \in \mathcal{N}_j$.*

3) Apply $\boldsymbol{u_i(k)}$ as its actual control input at instant k. Wait one time step, increment k, go to step 1).

Remark 2: *Though all subsystems update plan in Algorithm 1, only a sole subsystem i_k optimizes at a time instant k. All other subsystems $i \neq i_k$ renew their current plans, by taking the remanding part of the old optimal input sequence $\boldsymbol{u_i^*}$ and augmenting with the terminal control law $\boldsymbol{u_i^T}$, denoted by $\hat{\boldsymbol{u}}_{\boldsymbol{i}}\boldsymbol{(k)}$. The order in which all subsystems are optimized is dependent on the update sequence $\{i_1, \cdots, i_k, i_{k+1}, \cdots\}$, which is a pre-determined or dynamic sequence, or may be chosen by designer.*

Theorem 1: *Assume that there exists an initial solution in Step 0) of Algorithm 1, and that Assumption 1 is satisfied. Then, the DMPC input $\boldsymbol{u_i(k)}$ obtained from Algorithm 1 is recursively feasible, for each subsystem $i \in \mathcal{I}$. Furthermore, the overall network system is asymptotically stable.*

Proof. We first prove recursive feasibility of Algorithm 1. Now suppose that there exists a feasible solution to Problem $\mathcal{P}_i$ at instant $k-1$ for all subsystems $i \in \mathcal{I}$, such that also the Assumption 1 is satisfied.

First, consider the subsystem(s) i with $i \neq i_k$ at the following time instant k. The renewed plan $\hat{\boldsymbol{u}}_{\boldsymbol{i}}\boldsymbol{(k)} := \{\boldsymbol{u_i^*(k|k-1)}, \cdots \{\boldsymbol{u_i^*(k+T-2|k-1)}, \boldsymbol{k_i^T}(\boldsymbol{x_i^*(k+T-1|k-1)}, \hat{\boldsymbol{x}}_{\boldsymbol{j}}\boldsymbol{(k+T-1|k-1)})\}$ satisfies the input constraint (6d), since it takes the remanding part of the old optimal input sequence and augmenting with the terminal control law. Similarly, the corresponding state $\hat{\boldsymbol{x}}_{\boldsymbol{i}}\boldsymbol{(k)}$ satisfies the state constraint (6c). As for the coupling constraints (6e), both $\boldsymbol{x_i(k+t+1|k)}$ and $\{\boldsymbol{x_j}\}_{j\in\mathcal{N}_i}\boldsymbol{(k+t+1|k)}$ for $t \in \{0, 1, \cdots, T-2\}$, consisting of the remanding part of the old optimal state sequences, satisfy the coupling constraints due to the assumption that they satisfied at time instant $k-1$. For $t = T-1$, the coupling constraints are also fulfilled according to (6b), since $\boldsymbol{x_i(k+T|k)}$ lies inside the terminal region $\mathcal{X}^T$.

Then consider the subsystem i with $i = i_k$. It is obvious that the optimal control input $\boldsymbol{u_i^*(k)}$ and the corresponding optimal state $\boldsymbol{x_i^*(k)}$ obtained from optimizing the Problem $\mathcal{P}_i$ satisfy the constraints (6b)-(6e). Hence, the DMPC input $\boldsymbol{u_i(k)}$ obtained from the Algorithm 1 for all subsystems $i \in \mathcal{I}$ is a feasible solution.

Then we continue to prove that the overall network system is asymptotically stable.

Consider the Lyapunov function candidate $V(k) := \sum_{i=1}^N J_i(\boldsymbol{\varepsilon_i(k)}, \boldsymbol{u_i(k)})$ at time instant k. And the derivative of the Lyapunov function is shown as follows:

$$\begin{aligned} &V(k+1) - V(k) \\ &= \sum_{i\in\mathcal{I}\setminus\{i_k\}} J_i(\hat{\boldsymbol{\varepsilon}}_{\boldsymbol{i}}\boldsymbol{(k+1)}, \hat{\boldsymbol{u}}_{\boldsymbol{i}}\boldsymbol{(k+1)}) \\ &+ J_{i_k}(\boldsymbol{\varepsilon_i^*(k+1)}, \boldsymbol{u_i^*(k+1)}) - \sum_{i\in\mathcal{I}} J_i(\boldsymbol{\varepsilon_i(k)}, \boldsymbol{u_i(k)}) \\ &\leq \sum_{i\in\mathcal{I}} J_i(\hat{\boldsymbol{\varepsilon}}_{\boldsymbol{i}}\boldsymbol{(k+1)}, \hat{\boldsymbol{u}}_{\boldsymbol{i}}\boldsymbol{(k+1)}) \\ &- \sum_{i\in\mathcal{I}} J_i(\boldsymbol{\varepsilon_i(k)}, \boldsymbol{u_i(k)}) \\ &= \sum_{t=0}^{T-1} (\|\boldsymbol{\varepsilon_i(k+1+t|k+1)}\|_{Q_i}^2 \end{aligned}$$

$$
\begin{aligned}
&+\|\boldsymbol{u_i}(\boldsymbol{k}+1+\boldsymbol{t}|\boldsymbol{k}+1)\|^2_{R_i})\\
&+\|\boldsymbol{\varepsilon_i}(\boldsymbol{k}+1+\boldsymbol{T}|\boldsymbol{k}+1)\|^2_{P_i}\\
&-\sum_{t=0}^{T-1}(\|\boldsymbol{\varepsilon_i}(\boldsymbol{k}+\boldsymbol{t}|\boldsymbol{k})\|^2_{Q_i}+\|\boldsymbol{u_i}(\boldsymbol{k}+\boldsymbol{t}|\boldsymbol{k})\|^2_{R_i})\\
&-\|\boldsymbol{\varepsilon_i}(\boldsymbol{k}+\boldsymbol{T}|\boldsymbol{k})\|^2_{P_i})\\
&=\sum_{i\in\mathcal{I}}(\|\boldsymbol{\varepsilon_i}(\boldsymbol{k}+1+\boldsymbol{T}|\boldsymbol{k}+1)\|^2_{P_i}\\
&+\|\boldsymbol{\varepsilon_i}(\boldsymbol{k}+\boldsymbol{T}|\boldsymbol{k})\|^2_{Q_i}\\
&-\|\boldsymbol{\varepsilon_i}(\boldsymbol{k}+\boldsymbol{T}|\boldsymbol{k})\|^2_{P_i}+\|\boldsymbol{u_i^T}(\boldsymbol{k}+\boldsymbol{T}|\boldsymbol{k})\|^2_{R_i}\\
&-\|\boldsymbol{\varepsilon_i}(\boldsymbol{k}|\boldsymbol{k})\|^2_{P_i}-\|\boldsymbol{\varepsilon_i}(\boldsymbol{k}|\boldsymbol{k})\|^2_{Q_i})\\
&\leq\sum_{i\in\mathcal{I}}(-\|\boldsymbol{\varepsilon_i}(\boldsymbol{k}|\boldsymbol{k})\|^2_{P_i}-\|\boldsymbol{\varepsilon_i}(\boldsymbol{k}|\boldsymbol{k})\|^2_{Q_i})\leq 0 \qquad (12)
\end{aligned}
$$

where for the first inequality we used the fact that subsystem i_k optimized at time instant k, thus the associated energy decreases. And by the Step (7c) in Assumption 1, the second inequality is also established. Hence the overall network system is asymptotically stable. □

4 Numerical examples

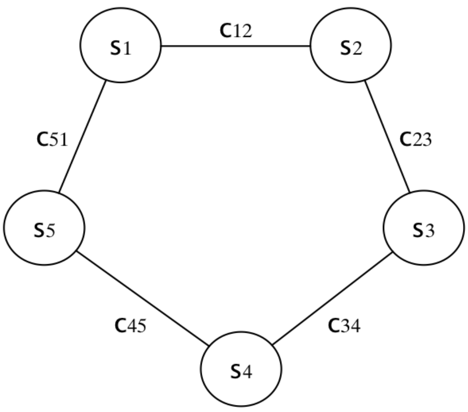


Fig. 2 *Five identical linear oscillators with coupled constraints*

The example in Fig. 2 considers consensus and synchronization of five identical linear oscillators s_i with c_{ij}, $i, j \in \{1, 2, 3, 4, 5\}$ indicating associated coupled constraints. Despite its simplicity, this example is of significant interest since it contains almost all the ingredients of a distributed control problem: dynamic coupling between subsystems (each oscillator and its cooperative set), modularity, and collective behaviors.

The dynamic of each oscillator is given by (1) with $\boldsymbol{A} = \left[\begin{smallmatrix} 0.9762 & 0.2169 \\ -0.2169 & 0.9762 \end{smallmatrix}\right]$ and $\boldsymbol{B} = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. Each input is constrained locally as $\|\boldsymbol{u_i}(\boldsymbol{k})\| \leq 1$. State constraints are defined by $\mathcal{X}_i = \{\boldsymbol{x_i} : \|\boldsymbol{x_i}\| \leq 10\}, \forall i$. And the weighting matrices for the state cost and control cost in (6a) are chosen as $Q_1 = Q_2 = Q_3 = Q_4 = Q_5 = I_2$ and $R_1 = R_2 = R_3 = R_4 = R_5 = 1$, respectively. For each subsystem, denote by the 2-nearest neighbors and itself as the cooperative set. The terminal control gain $\boldsymbol{k_i^T}$ and terminal state matrices P_i are calculated by solving the LMI (11) through LMI tools in Matlab.

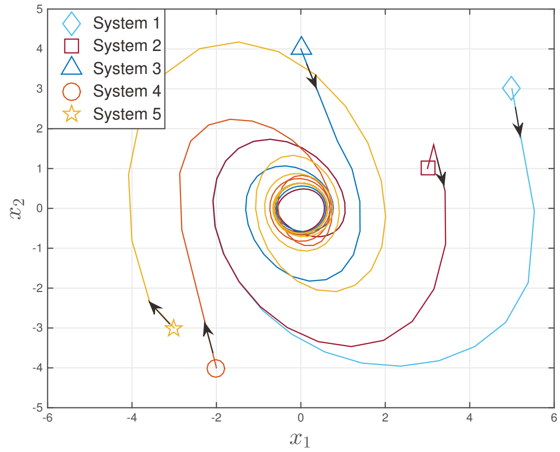
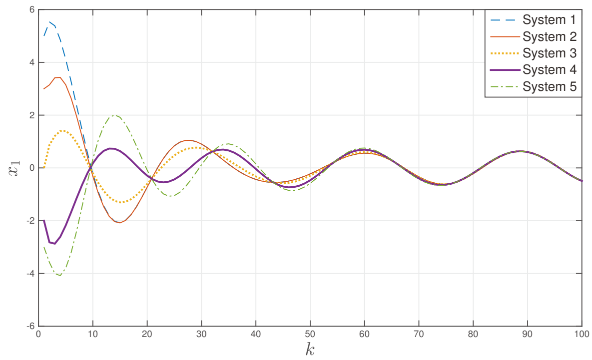
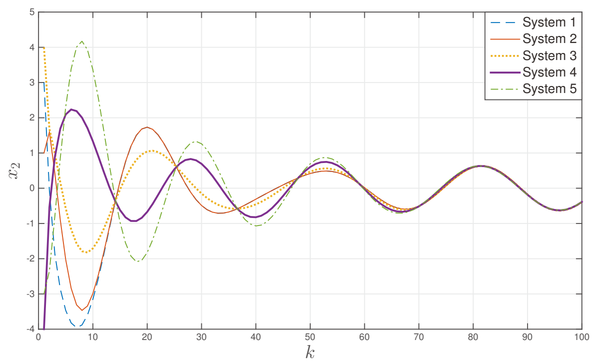


Fig. 3 *State trajectories - phase plane*



(a) *x_1 - time domain*



(b) *x_2 - time domain*

Fig. 4 *State trajectories - time domain*

Fig. 3 shows the state trajectories in phase plane with prediction horizon $T = 10$. Five oscillators, starting from five different positions, asymptotically achieve the same harmonic vibration after certain steps. They together accomplish the cooperative tasks like consensus and synchronization problems different from the stabilization of an a priori known set point which most previous work has considered. State trajectories in time domain drawn in Fig. 4 also illustrates that five linear oscillators asymptotically achieve the consensus and synchronization.

5 Conclusion

In this paper, we have presented a DMPC strategy with consideration of the error of state between each subsystem and the center of its cooperative set to handle the cooperative tasks like consensus and synchronization of linear systems with coupled constraints. And only one subsystem optimize, while other subsystems renew their plans at each time instant in order to guarantee the robust feasibility of the DMPC algorithm. The impact on the linear systems of the DMPC strategy has also been analyzed sufficiently and verified illustratively.

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